Travel probability fields and urban spatial structure:  
1. Theory

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Received 16 December 1980; in revised form 13 May 1982

Abstract. The proposed model attempts to explain how particular spatial distributions of trip destinations might arise as manifestations of the accessibility benefits and travel costs associated with a housing location. The trip distributions are elliptical, being expressed as bivariate normal distributions. The parameters of these distributions are shown to be related to the parameters of an assumed density function for activity sites (such as jobs and shops) and to travel speeds. The model implies a housing density function in a monocentric urban area which is negative exponential.

1 Introduction

Models of person movement within urban areas have generally used the concept of individual trips. Examples include the familiar gravity, intervening opportunities, and entropy-maximization models frequently employed for traffic analyses in transportation planning studies, and also the complex disaggregate models developed recently to describe trip generation, trip distribution, and mode split.

It has been persuasively demonstrated that many important causal factors in travel demand are not present in such trip-based models. Chapin (1965), Hìgerstrånd (1976), Zahavi (1974), Cullen and Godson (1975), Burn (1979), Damm (1979), Jones (1979), and others have argued that it is necessary to abandon the trip-based approach in favor of an approach involving total household activity patterns.

The research described here aims to explain interrelationships between travel and urban structure through the use of measures of total household travel which are independent of how individual trips are defined, linked, or otherwise combined (Zahavi et al. 1981). These travel measures include total daily travel distance, total daily travel time, and expenditure, and the geographic distribution of activity sites visited outside the household. The present paper focuses on the theoretical aspects of the research. Empirical aspects will be presented in a subsequent paper (Beckmann et al. 1983).

2 Background

Zahavi (1979) showed that the spatial distribution of activity sites visited by households in a particular residential location can be depicted by a probability density function described in locational coordinates. Certain properties of the density functions were shown to be related to the residential location relative to the urban center(s), to the socioeconomic characteristics of the household, and to travel speeds and costs. Zahavi showed that bivariate normal distributions positioned at one standard deviation (or another isoprobability contour) are effective representations of the spatial distributions of trip destinations. By use of examples computed for the Washington, DC, USA (figure 1) and Nuremberg, FRG (figure 2) metropolitan areas, it was shown that: (1) the major axis of the ellipse tends toward an urban center; (2) the ellipses are more elongated, the farther the origin households are
Figure 1. Travel probability fields in 1968 for selected districts in Washington, DC. Source: Zahavi, 1982.

Figure 2. Travel probability fields for car travel for selected districts in the Nuremberg region in 1975. Source: Zahavi, 1979.
located from the urban center; (3) car travel fields are more elongated than transit travel fields for the same households; and (4) the direction of the major axes of the ellipse is also affected by available supply of transportation systems such as bus routes.

Zelinsky's work is consistent with attempts made in the field of urban geography to describe the spatial distribution of activity patterns (Chapin, 1965). Moore (1970) proposed a probabilistic approach in which trips result from the likelihood of contact between two fields, expressed by a joint probability function, with trip distances and travel opportunities represented as component marginal distributions. This approach also suggests the possibility of bivariate normal distribution of activities, namely that trip density rings would be elliptical.

Related efforts are to be found in the works of Angel and Hyman (1972; 1976), Wilkins (1969), Vaughan (1974), and Blumenfeld (1977). Angel and Hyman (1972) derived the probability density of commuter traffic in Manchester as a continuous function of distance from the urban center. Inputs to the model included density functions for the locations of homes and workplaces, and a function describing travel speeds as a function of distance from the urban center (Angel and Hyman 1970). Trip distribution was accomplished using entropy maximization. Wilkins (1969) calculated theoretically the probability density function for the lengths of commuter trips given bivariate normal distributions for the densities of homes and workplaces described in the two-dimensional spatial plane. Calculations were performed for the different types of road network geometries used by Smed (1963), Haith (1964), and Tan (1966) in deriving probability density functions for commuter travel into central areas of cities. Wilkins's calculations were extended by Vaughan (1974) to account for correlations between the distributions of homes and workplaces. Blumenfeld (1977) developed a continuous trip-distribution model using the Wilkins-Vaughan approach, and Khoo and Vaughan (1979) provided calculations for many additional probability measures associated with the spatial distributions of travel.

These previous results hold forth a promise that travel patterns might be effectively related in an activity-behavior sense to urban spatial structure. Toward that end, a model of residential location and travel generation is proposed here. The model is intended to be consistent with the mathematical urban-economic models of residence and activity location pioneered by Alonso (1964), Mills (1967; 1969), Beckmann (1966), Mathi (1969), and others. Such models are reviewed in Mills and MacKinnon, 1973; Atlas and Dendiras, 1976; and Richardson, 1977.

3 Geometric interpretation
The present model is specified for homogeneous groups of households, each group defined by locational proximity, income, car ownership, and possibly other socioeconomic strata. It is postulated that the trips generated by such a homogeneous group are influenced by a small number of discrete urban centers. Each center defines a spatial distribution of trip attractors. Regardless of the number of centers, they can be considered hierarchically (Papageorgiou and Caselli, 1971). The highest-order center probably accounts for the greatest number of trips for the household group; presumably it is a central business district (CBD).

It is appropriate to consider only the highest-order center for the initial model specification. In further, more elaborate, specifications of the model, it might be possible to account for multiple centers, following a theoretical approach such as that of Hartwick and Hartwick (1974), White (1976), Romanos (1976), or of Ollard (1976), among others.

The highest-order center can be interpreted either as the location of all household employment, which is consistent with earlier urban-economic models, or as the focus of economic opportunities and related reasons for agglomeration.
location and an angle $\phi$ to the vertical given by $\tan \phi$, which is equal to some measure of the speed of the transportation system. Such a geometric depiction of time and space has been suggested by Chapin (1965) and developed in detail by Hagerstrand (1970). It has been adopted and extended in analyses of accessibility by Lenntorp (1970, 1976) and Burns (1979).

If the cone of travel possibilities (figure 4) and the surface describing density of attractors (figure 3) are superimposed, the intersection of the two surfaces circumscribes an approximately elliptical area on the density of attractors surface, as shown in figure 5. The analytics of the geometry are greatly simplified by approximating the exponential density surface by a tilted plane in the neighborhood of the household location, also as shown in figure 5. Such an approximation is reasonable in view of the smoothness of an exponential surface. The approximation becomes better the farther away from the urban center the household is located, which reinforces the fact that the present model is inappropriate to household locations in the vicinity of the center. Moreover, the approximation is likely to be more accurate for large urban areas than for small ones, since it implies that there is a negligible decline in travel attractions in the neighborhood of the household in a direction transverse to the urban center. (The approximation is merely a point of mathematics, not an additional economic assumption.)

The intersection of the surfaces is then precisely an ellipse. It is proposed that the volume within the cone under the ellipse is related to the travel benefits which can be realized by a household located at $r_0$. The geometric interpretation of these travel benefits is facilitated by referring to figure 6.

Consider a trip from the household location at $(x_0, y_0)$ to any point $(x_1, y_1)$ in the $x-y$ plane. The vertical projection $A'$ from $(x_1, y_1)$ to the tilted attractions plane represents the utility or gross benefits attributed to the trip. (That is, the activities at the trip destination are the utility or gross benefits associated with the trip.) The vertical projection $C'$ from $(x_1, y_1)$ to the surface of the travel-time or disutility cone represents the costs attributed to the trip. The projection $B' = A' - C'$ represents the net benefits of the trip. The integral of all such trips with $B' > 0$ is the volume of the cone beneath the tilted plane, the total net travel benefits associated with residential location $(x_0, y_0)$. Trips to points outside the ellipse will not be made because the costs of such trips are greater than the gross benefits. At the boundary of the ellipse costs equal benefits, and the longest trips to the boundary are in the direction of the urban center.

**Figure 5.** The intersection of the cone of travel possibilities and the surface describing density of attractors.

**Figure 6.** Geometric interpretation of travel benefit.
Such a representation of net benefits is essentially a three-dimensional analog to
consumer surplus. Through this representation, calculations of travel benefits as
functions of the parameters of the spatial distribution of travel attractions and travel
efficiency functions can proceed by use of the analytic geometry of cones and right
circular cones. The parameters of the former include the density of activity sites at
the urban center and the rate at which activity-site density decreases with distance
from the center; the single parameter of the latter is a measure of travel speed. All
of the calculations are relative to household distance from the urban center.

The initial specification of the model fails to consider money costs of travel,
differences among households with regard to types of activity sites preferred, and
potential asymmetry of travel networks (directional differences in speed) and of
activity-site densities. Such considerations are the subjects of possible further
research, as outlined in section 7.

4 Accessibility benefits

Figure 7 depicts a cross section through the urban center and household location.

Equations for the axes of the ellipse formed by the intersection of the plane and
the cone are:

\[ \alpha = \frac{ak}{k^2 + b^2} \]  \hspace{1cm} (1)

and

\[ \beta = \frac{a}{(k^2 + b^2)^{1/2}} \]  \hspace{1cm} (2)

where

- \( a \) is the level of attractions at the household location,
- \( b \) is the slope of the attraction distribution,
- \( k \) is the slope of the cone of travel possibilities (inverse of travel speed),
- \( \alpha \) is the major axis of the ellipse which is oriented along the ray from the urban
center and
- \( \beta \) is the minor axis of the ellipse which is parallel to the pivot line through the
household's focus.

Furthermore, the minimum distance from the household-location point to the plane
of the attraction distribution is given by

\[ h = \frac{a}{(\beta^2 + 1)^{1/2}} \]  \hspace{1cm} (3)

a quantity derived using the cosines and tangents of the angle \( \phi \) in figure 7.

![Figure 7: A cross section through the urban center and household location.](image-url)
Thus, the equation for the net travel benefits at distance $x$ from the origin is

$$B = \frac{\gamma - \delta}{(\alpha - \beta)^2 + (\gamma - \delta)^2}.$$

And with the introduction of linear approximation for travel time and travel benefits yields

$$\frac{B}{\gamma - \delta} = \frac{B}{(\alpha - \beta)^2 + (\gamma - \delta)^2}.$$

And with the introduction of the linear approximation for attractions as a function of the opportunity, $\alpha$, at the urban center.

Rearrangement of equation (6),

$$r = \frac{1}{B(\alpha - \beta)^2 + (\gamma - \delta)^2}.$$

Here it is assumed that trip frequency does not vary significantly so that there is a linear relationship between mean trip distance and travel time expended in travel. Equation (5) relates the mean distance to the elliptical travel area.

From the volume of the cone beneath the plane $z = 0$, provided that the ratio of $A$ to $B$ does, that the density of the travel attractions is constant. This condition must be satisfied in the case of urban attractions.

Thus the equation for the net travel benefits at distance $x$ from the origin is

$$B = \frac{\gamma - \delta}{(\alpha - \beta)^2 + (\gamma - \delta)^2}.$$
satisfied for exponential density functions at locations sufficiently far from the urban center. At locations near the center it is not expected to be satisfied. Thus, for exponential density functions, travel fields implied by the model might become less elongated, reach a minimum elongation, then become more elongated as distance increases from the urban center. Last, equation (10) implies that the travel ellipses for car travel are more elongated than those for transit travel, if the speed of car travel is greater than the speed of transit travel, if we assume that speeds are the same in all directions from the household. This might not be the case for transit travel because radial routes might have better service than transverse routes. In such a case, the circular cone could be replaced by an elliptical cone, but this complicates the analysis and is a subject for future research.

5 An alternative random utility-model

The elliptical travel fields in the prior approach can also be generated through application of a random utility model of household trip-making behavior. This random utility model reinforces the prior approach by clarifying certain analytical points not readily derivable in the prior analysis, and directly addresses the definition of travel-probability fields as isoprobability contours of binomial normal distributions functions.

Consider a household located at any point in the urban space. It is convenient to define this arbitrary point as the pole in a polar coordinate system \((r, \theta)\) where the coordinate angle \(\theta\) is measured relative to the vector connecting the household location and the urban center, as shown in figure 8.

With the assumption once again that the negative exponential density function for travel opportunities can be approximated in the neighborhood of the household location by a plane tilted in the direction of the urban center, the level of economic opportunities at trip destination \((r, \theta)\) is

\[
\eta = a - b r \cos \theta .
\]  

(11)

From a deterministic viewpoint, the decision of the household whether or not to make a trip to destination \((r, \theta)\) depends upon whether or not travel benefits, attributed to the opportunities at \((r, \theta)\), are greater than the cost of the trip. A trip is made whenever

\[
\eta > kr ,
\]  

(12)

that is, whenever

\[
a \cdot br \cos \theta > kr .
\]  

(13)

or whenever

\[
r < \frac{a}{k} \left( 1 + \frac{b}{k} \cos \theta \right) .
\]  

(14)

Now the critical distance at which cost equals benefits is given by

\[
r = \frac{a}{k} \left( 1 + \frac{b}{k} \cos \theta \right) .
\]  

(15)

Figure 8. The polar coordinate system \((r, \theta)\) used to describe the locations of the urban center and the household.
which is the equation of an ellipse in polar coordinates with the pole (household location) at one focus and eccentricity given by

\[ e = \frac{b}{a}. \]  \hspace{1cm} (16)

Since the eccentricity, \( e \), of the ellipse is defined as

\[ e = \frac{(a^2 - \beta^2)^{1/2}}{a}, \]  \hspace{1cm} (17)

where \( a \) and \( \beta \) are the major and minor axes, respectively, equation (16) can be compared with the eccentricity of the ellipse derived in the previous model. If both sides of equation (17) are squared

\[ e^2 = 1 - \frac{\beta^2}{a^2}; \]  \hspace{1cm} (18)

\( b/k \) is substituted for \( e \) [equation (16)] and equation (18) is rearranged then.

\[ \frac{a}{\beta} = \left(1 - \frac{b^2}{a^2}\right)^{1/2}. \]  \hspace{1cm} (19)

A comparison of equations (10) and (19) shows that the results are identical in either geometric specification.

Random components might be associated both with the density of economic opportunities, as reflected in the locations of trip attractors, and with the perception of the household of travel benefits and costs. First, let \( q \), where

\[ q = q(r, \theta) dr d\theta, \]  \hspace{1cm} (20)

denote the probability that a trip attractor is located in the differential area centered at trip destination \((r, \theta)\) at distance \( r \) from the household. Second, introduce a random component of utility, \( \epsilon \), assumed to be distributed independently of \( q \). The decision to make a trip to \((r, \theta)\) then depends upon whether benefits are greater than costs, taking into account the random utility component and the probability of finding a trip attractor, \( P \)

\[ P = q_p(u + \epsilon - kr > 0), \]  \hspace{1cm} (21)

or

\[ P = q_p(\epsilon > kr - u + hr \cos \theta). \]  \hspace{1cm} (22)

Isoprobability contours for trip destinations are then defined by

\[ kr - u + hr \cos \theta = C_0, \]  \hspace{1cm} (23)

where \( C_0 \) is a constant. This is the equation of an ellipse in the polar coordinate system

\[ r = \frac{(C_0 + u)}{k(1 + h \cos \theta)}. \]  \hspace{1cm} (24)

To demonstrate isoprobability contours, it is usually assumed that a random component such as \( \epsilon \) is distributed normally. Equation (21) for the probability of making a trip to a particular destination becomes

\[ P = q \left[1 - N\left(\frac{kr - u - \mu}{\sigma}\right)\right], \]  \hspace{1cm} (25)

where \( N() \) denotes the standardized normal density function with mean \( \mu \) and
standard deviation \( \sigma \). However, the forms of the isoprobability contours for equation (16) are more apparent when the normal distribution is approximated by a logistic:

\[
N(t) \approx \frac{1}{1 + \exp(-\delta t)} .
\]

(26)

where Nash (1976) has shown that for the value of \( \delta = 1.70174 \)

\[
\left| N(t) - \frac{1}{1 + \exp(-\delta t)} \right| < 0.00946 .
\]

(27)

to all \( t \), and that value of \( \delta \) minimizes the bound.

Thus,

\[
P \approx \frac{\eta}{1 + \exp(\delta/\sigma)(kr - \mu - \mu)} .
\]

(28)

and it is clear that the isoprobability contours are described by constant values of the exponent:

\[
\frac{\delta}{\sigma}(kr - \mu - \mu) = C_1 .
\]

(29)

where \( C_1 \) is a constant; or, on rearrangement of terms and substitution of expression (11) for \( u \),

\[
r = \left( \frac{\eta C_1}{\delta} + \mu + \mu \right) / k \left( 1 + \frac{\delta}{\sigma} \cos^2 \theta \right) .
\]

(30)

which is once again the equation of a ellipse. The shape of the ellipse is determined by the parameter \( k/\eta \) defining the eccentricity or elongation, and for given values of \( \mu, \sigma, \delta, \) and \( \eta \), the constant \( C_1 \) determines the area. Conversely, the constant \( C_1 \) can be set so that the isoprobability contour corresponds to a particular volume under the probability density function.

Equation (28) shows that, with the direction defined by \( \theta \) held constant, trip frequency falls off with distance as \( 1 - L(r) \), where \( L(r) \) is a logistic function. For large \( r \), trip frequency is approximately exponential:

\[
P \approx \eta \exp \left[ -\frac{rb}{\sigma}(k - b + b \cos \theta) \right] .
\]

(31)

The coefficient in the exponent, the rate of decline, is greatest when \( \theta = 0 \), that is, in the direction away from the urban center. It is smallest when \( \theta = \pi \), that is, in the direction toward the urban center.

Consequently, it is appropriate to employ probability distribution theory to analyze the travel ellipses revealed in travel survey data on trip distributions. If shorter length trips are more frequent than longer trips for the same trip purpose, two-dimensional bell-shaped probability distributions, such as the bivariate normal distribution, would be consistent with the elliptical model in that concentric ellipses represent isoprobability contours for these distributions. A test of the model for a given urban environment and household group would involve analysis of the parameters of the distributions fit to the observed travel data.

6 Decisions of household location
The initial model proposed for choice of household location assumes that households have utility functions with three components dependent on residential location: housing, travel benefits (that is, accessibility), and (negative) housing cost. Additional
utility components will not affect residential location if they are separable with respect to the spatial components.

Utility for a household located at distance \( r \) from an urban center is thus given by

\[
U = H + B - pq,
\]

where

- \( H = H(q) \) is the utility of housing space, \( q(r) \),
- \( B = B(r) \) is the travel benefits obtainable at location \( r \), and
- \( p = p(r) \) is the price of housing at \( r \).

Here it is assumed that the utility of general consumption is separable and linear.

With the assumption that the utility of housing space is logarithmic as in the standard urban-economic model of residence location (Beckmann, 1974):

\[
U = \gamma \ln q + B - pq.
\]

Income enters the present form of the proposed model as a stratification variable, and car ownership can be treated likewise, as might other types of household characteristics. (The subscript \( i \), referring to household stratum \( i \), has been dropped from the utility components and overall utility level for reasons of simplicity.)

The household then chooses its housing such that

\[
\max_{q(r)} U = \max_{q(r)} \{ \gamma \ln q + B - pq \}.
\]

The necessary and sufficient condition for such a utility maximum is

\[
g = \frac{\gamma}{p}.
\]

and achieved utility \( U^* \) is found by substituting the utility maximizing solution (35) into the utility function (33):

\[
U^* = \gamma \ln \frac{2}{p} + B - \gamma.
\]

With expression (9) for approximate travel benefits, achieved utility becomes

\[
U^* = \gamma \ln \frac{2}{p} - \gamma + \frac{3r^2A - br}{4k^2(1 + b^2)^{1/2}}.
\]

The solution for housing price, \( p(r) \), is

\[
p(r) = \gamma \exp \left[ - \frac{U^* - \frac{3r^2A - br}{4k^2(1 + b^2)^{1/2}}}{\gamma} \right].
\]

Equation (38) describes the rent–bid function for the particular socioeconomic stratum of households in question. Since achieved utility \( U^* \) must be constant in the definition of the rent–bid function, equation (38) can be written as

\[
p(r) = C \exp \left[ - \frac{3r^2br}{4k^2(1 + b^2)^{1/2}} \right] = C \exp(-\lambda r),
\]

where \( C \) and \( \lambda \) represent spatially independent parameters at first approximation. Thus, the rent–bid function implied by the proposed residential location model is a negative exponential in terms of distance \( r \) from the urban center.

Furthermore, residential density is also a negative exponential function of distance:

\[
\frac{1}{q(r)} = \frac{p(r)}{\gamma} = C \exp(-\lambda r),
\]

(40)
which implies that residential density and the density of economic opportunities are log-linearly related.

Results (39) and (40) are consistent with results obtained in a different fashion in urban-economic models proposed by Muth (1969), Mills (1969), Hines (1970), Papageorgiou (1971), and others. These results demonstrate how activity-site densities and travel efficiencies can be interrelated with housing rents and densities.

7 Conclusions and suggestions for further research

Three major conclusions can be drawn from the research reported here:

1. Empirically developed fields of travel probability can be shown to be the expected result of household travel behavior which takes into account the location of travel opportunities (activity sites) and travel impedances (locational dependent travel speeds).

2. A theoretical measure of location accessibility benefits can be derived from the activity-site density function and travel speed function. This measure is a predictor of the shape of the resulting fields of travel probability for any household location.

3. Application of the accessibility-measure in a residential location model leads to results, such as exponential rent-bid functions, which are consistent with results obtained in the standard mathematical model in modern urban economics.

Five suggestions for further theoretical research follow from these conclusions:

1. The model might be extended to account for asymmetries in activity-site densities due to multiple urban centers. Focus in such research would be on approximations for circumventing complicated analyses.

2. Similarly, the model might be extended to account for asymmetries in travel speeds resulting from network geometries and directionally dependent traffic congestion.

3. Money costs of travel could be included by defining a composite travel disutility comprised of elements both of money and of time. Alternatively, the concept of generalized costs might be used, but this masks explicit trade-offs between time and money.

4. Households might be segmented on the basis of differences in activity preferences, with separate activity-site density functions for each segment. This would address one important aspect of residential location variations by household sociodemographics.

5. Finally, a dynamic version of the model might be developed to incorporate urban growth and effects of changes in travel variables. This is the most difficult subject area, but it is perceived to be the most important one in terms of model forecasting ability.

Acknowledgements. The authors are grateful to Robert W. Crash and David Kahn, of the US Department of Transportation Research and Special Programs Administration, for their permission to refer to results from a study conducted for their agency. Appreciation is also extended to David Boyle and an anonymous referee for their constructive comments. However, the responsibility for the views expressed in this paper rests solely with the authors.

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